Semi Unit Graphs of Commutative Semi rings

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Abstract

We explore graphs relative to semi unit elements in commutative semirings and study the characteristics of these graphs.

1 Introduction and Preliminaries

The concept of semirings was introduced by H. S. Vandiver in 1935 and has since then been studied by many mathematicians with different angles. The concepts of the graph to the element (zero-divisor) of a ring was introduced by Beck in [17] and discussed the coloring of a commutative ring. The zero-divisor graph of a commutative ring has been studied extensively by several authors like D. F. Anderson , P. S. Livingston and Shahabaddin Ebrahimi Atani in [8,9,11]. But S.E Atani and his colleagues establish the propertes of zero divisor graphs linked with semirings .He also introduced unit graph and some other useful graphs in semirings in [9,11].In this paper, We focus our work on a special element of semiring called semi unit and our work is relevant with its some results and mainly based on study of its graphs.
For the sake of completeness, we state some definitions and notations used throughout. A non-empty set S is called semi ring, if it is semi group under two binary operations addition and multiplication as well ,and these two binary operations are linked with distributive law. Golan[2] includes the additive identity zero also in semi ring. A Semiring S is commutative if $ab = ba$ for all $a, b \in S$ and this commutative semiring S is commutative with identity if there exist $1 \in S$ such that 1.a = a.1 for all $a \in$ S . In this paper we utilized the semi ring as commutative semi ring with non-zero identity else mentioned otherwise. A non-zero element a of S is said to be unit in S if there exist a non-zero element a of S is said to be unit in S if there exist $0 \neq r \in S$, such that r.a= a.r = 1.
Let $S \neq \phi$, be a commutative semi ring with non-zero identity. A non-zero

element a of S is said to be semi unit in S if there exists, $r, s \epsilon S$ such that $1 + r.a = s.a$. The set of all semi units of S will by denoted by S_u and the set of all non-semi units will be denoted by N_u in this paper.Every unit is a semi unit as by taking r=0.In a ring every semi unit is a unit.A non-empty subset I of S is called an ideal of S if$a, b \in \ I$ and $r \ \in \ S$ then $a + b \ \in \ I$ and $r.a \ \ \in \ \ I$. A prime ideal of semi ring S is a proper ideal P of S in which $x \epsilon P$ *or* $y \epsilon P$ *whenever* $x.y \epsilon P$. A proper ideal M of semi ring S is said to be maximal , if M is an ideal in S such that $M \subsetneq J$, then J=R. Every Maximal ideal is prime if S is commutative semiring with unity [6]. A Subtractive ideal (k-ideal) K is an ideal (*may not proper*) such that if $x, x + y \ \in K$ then $y \ \in K$. 0 is a k-ideal of semi ring S. A k-ideal which is also maximal ideal is called Maximal k-ideal .if P is maximal k-ideal of S if and only if S/P is a semi field [6]. Let P be an ideal of a semi ring S , P is a prime k-ideal of S if and only if S/P is a semi domain. Let S be a semi ring with non-zero identity. S is said to be a local semi ring if and only if S has a unique maximal k-ideal. The k-Closure cl(I) of ideal I is defined by cl (I) ={ $a \epsilon S$; a + c = d for $c, d \epsilon I$ } is an ideal of S.

An ideal I of semiring S is called a partitioning ideal(Q-ideal) if there exists a subset Q of S such that: S =∪ {q +I : q ∈Q } . If $q_1, q_2 \epsilon Q$ then $(q_1 + I) \bigcap (q_2 + I) \neq \phi$ if and only if $q_1 = q_2$. If I is a partitioning ideal(Q-ideal) of a semiring S then I is a subtractive ideal (k-ideal) of S ,by (lemma 2[16]) .

A graph ξ consists of a vertex set $V(\xi)$ and an edge set $E(\xi)$, where an edge is an unordered pair of distinct vertices of ξ .The graph with no edges is the null graph. The number of vertices in a graph ξ is called its order, and number of edges are its size.The degree of a vertex v in a graph , denoted by d(v), is the number of edges of incident with v, each loop counting as two edges.A graph is regular if degree of every vertex are same. If every vertex is adjacent with n other vertices than this graph is called n-regular graph.The distance between two vertices of a graph ξ is the number of edges among the shortest path of these two vertices . If there is no path then the distance is taken as infinity. The diameter of a graph ξ is the greatest distance between two vertices of ξ .

There are some special families of graphs as complete graph is a simple graph in which any two vertices are adjacent.A graph is connected if there is a path between any two vertices.A graph is totally disconnected if every two vertices are non-adjacent. A complete graph is closed if each pair of vertices is joined by every edge (loops included) that if there is loop then every vertex must have loop. If not than no vertex has loop. A graph is bipartite if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y .If ξ is simple and every vertex in X is joined

to every vertex in Y, then ξ is called a complete bipartite graph. We write $K_{m,n}$ to mean a complete bipartite graph with m vertices in one set and n in the other.A star is a complete bipartite graph ξ with $|X| = 1 or |Y| = 1$. A cycle that passes through every vertex in a graph is called a Hamilton cycle and a graph with such a cycle is called Hamiltonian.An isomorphism between two simple graphs X and Y as a bijection $\theta :$ V (X) $\longrightarrow$ V (Y) which preserves adjacency .

2 Results of Semi Units

2.0.1 Lemma[6]

Let S be a semiring and let $a \epsilon S$ Then a is a semi-unit of R if and only if a lies outside each k-maximal ideal of S.

2.0.2 Lemma[6]

If $x \epsilon S$ be a commutative semi ring , then cl (Sx) is a k-ideal of S.

2.1 Proposition

Every ideal generated by a non-semi unit in a commutative semiring S with identity is proper ideal and contain in some k-maximal ideal.

Proof Consider a be a non-semi unit in S. We will show that $\langle a \rangle$=Sa is a proper ideal. Firstly, we will prove that it is an ideal. Suppose on contrary that
a + a = 2a is semi unit then by definition of semi unit 1 + (2a).r = (2a).s for some $r, s \epsilon S$. implies that 1 + 2r.(a) = 2s.(a) tells that a is semi unit, which is contradiction so 2a is also non-semi unit .In general we can say that ma (m is any positive integer) is non-semi unit so is closed under addition. Now , by taking any $r \epsilon S$. we need to show that a.r is also non-semi unit .Suppose on contrary that a.r is semi unit ,then 1 + (a.r)t = (a.r)s for some $t, s \epsilon S$, 1 + a(r.t) = a(r.s) implies that a is semi unit, contradiction.Hence it is an ideal , also proper since $1 \notin \langle a \rangle$ Now we prove that it contains in some k-maximal ideal. Here $1 \in S$ therefore$\langle a \rangle = Sa$. Also $Sa \subseteq cl(Sa)$,which is k-ideal .so by (Corollary 2.2 [12]) Every k-ideal of semi ring contained in some k-maxiaml ideal,so there must exist some k-maximal ideal M (say) such that $\langle a \rangle = Sa \subseteq cl(Sa) \subseteq M$ implies that $\langle a \rangle = Sa \subseteq M$.

2.2 Proposition

The set of semi units S_u of semi ring S with non-zero identity makes a semigroup under multiplication.

Proof Consider a,b are any two semi units in semiring S with identity, such that 1 + r.a = s.a and 1 + t.b = u.b for some $r, s, t, u \epsilon S$ Now to check that a.b is semi unit .We suppose on contrary that a.b is a non-semiunt, therefore there exist a proper k-ideal$\langle ab \rangle$, so there must exist k-maximal ideal M such that $\langle ab \rangle \subseteq M$ implies that $ab \epsilon M$ .This tells that $sab, rab \epsilon M, for\ all\ r, s \epsilon S$. Also we have 1 + r.a = s.a $\Rightarrow b + r.a.b = s.a.b \epsilon M \Rightarrow b + r.a.b \epsilon M \Rightarrow b \epsilon M (since\ k - ideal)$ which is a contradiction because b is a semi unit.

2.3 Theorem

Let S be a semi ring with identity, provided that N_u is not ideal then N_u is not closed with respect to addition.

Proof It is given that N_u is not an ideal . Now, It is enough to show that is always closed under multiplication of its elements by elements of S.

Let $a \in S\ and\ x \in N_u$. We will show that $a.x \in N_u$. Suppose on contrary that $a.x \in S_u$. It gives

1 + (a.x).r = (a.x).s , for some $r, s \in S$. $\Longrightarrow 1 + a.(x.r) = a.(x.s) \Longrightarrow 1 + a.r_1 = a.s_1$ where $r_1 = x.r, s_1 = x.s \in S$ this implies that x is semi unit , which is contradiction ,so $a.x \in N_u$.Therefore, it is non-ideal only if there may some distinct $x, y \in N_u$ such that $x + y \in S_u$.

3 Semi Unit Graphs

Consider S is commutative semi ring and S_u denotes of set of all semi unit elements in S. We originate a semi unit graph ,denotes by ξ , by taking every distinct elements as vertices, such that if $x, y \in S$ then x and y are adjacent if $x + y \in S_u$. And there will be no edge if $x + y \notin S_u$. We shall discuss about its properties, characteristics and different shapes of Semi Unit graphs in this section.

3.1 Theorem

Consider a Semi ring S, then the Semi unit graph is complete if and only if $S_u = S - \{0\}$, provided S is not a ring.

Proof If $S_u = S - \{0\}$, then S_u is zero-sum free, since S is semi ring otherwise it will be ring (c.f. Lemma 2.1 [5]).This shows thatS_u is closed

under addition, so every two vertices of S_u are adjacent. Also 0 is adjacent with all other elements of S as $0 + x = x \epsilon S_u$, therefore graph is complete. Conversely, Suppose ξ is complete graph, every vertex $x \in S$ of graph is connected with all other vertices.Since $0 \in S$, therefore 0 is also adjacent to every $x \in S$ and
0 + x = x is semi unit $\Longrightarrow x \in S_u \ \forall\ 0 \neq x \notin S \Longrightarrow S_u = S - \{0\}$. .

3.2 Theorem

Consider a Semi ring S with non zero identity. If N_u makes k-ideal then ξ is connected graph.

Proof Consider S is semi ring with non zero identity also suppose that the set of all non-semi units makes k-ideal. Let $x, y \in N_u, then x + y \in N_u$, that there is no edge between them. Now consider $x \in N_u$, and $u \in S_u$ then
x + u must be semi unit, Suppose on contrary that $x \in N_u$,and $x + u \in N_u$ $\Longrightarrow u \in N_u$ which is contradiction , so $x + u \in S_u$, there is always an edge between a semi unit and non- semi unit in this semi ring, so all vertices must be connected at least through identity 1.Hence graph is connected.

3.2.1 Lemma [6]

Let S be a semiring. Then S is a local semiring if and only if the set of non-semi-unit elements of S is a k-ideal.

3.2.2 Corollary

The Semi unit graph of Local Semi ring S ($|S| \geq 2$), with identity 1 is always connected.

3.3 Theorem

If S be a commutative semi ring without identity, the semi unit graph is totally disconnected if and only if $S_u = \emptyset$

Proof Let ξ is totally disconnected graph, then 0 is also not adjacent to any vertex $x \in S$ which tells that no element of semi ring is semi unit. Conversely, Let $S_u = \emptyset$ and suppose on contrary that there exist elements $a, b \in S$, such that there is an edge between a and b, that a + b is semi unit, which is a contradiction .Hence graph is totally disconnected.

3.4 Proposition

Let S be a commutative semi ring then $1 \leq\mid \xi \mid< \infty$ (i.e. semiunit graph is finite) ,if and only if S is finite or not a semi field.

Proof For S is finite, then it is trivial.

If S is infinite, then $\mid \xi \mid$ is finite if number of semi units $|S_u|$ are finite. Also we know that $U(S) \subseteq S_u$ and here $\mid \xi \mid$ is finite which tells that S cannot be infinite semi field (otherwise every element is unit so also semi unit in semi field) .

Conversely, Suppose that S is infinite semi field then $S_u = U(S) = \infty$ then $\mid \xi \mid= \infty$ since 0 is adjacent with all semiunits (units). Thus $1 \leq\mid \xi \mid< \infty$ if and only if S is not a semi field.

3.5 Remarks

1- S be a semi ring with cancellative law and $S_u = \{1\}$ *with* N_u makes ideal then $\mid \xi \mid = 1$.

2- In a ring the unit graph and semi unit graph are isomorphic, since all semi units are units in ring.

3.6 Proposition

If two semi ring R, S are isomorphic then their graphs $\xi(R)$ and $\xi(S)$ are also isomorphic.

Proof Clearly,$|R| = |S|$, So, number of vertices are equal. Now, we will prove that the adjacency of vertices are also preserved. Firstly, we shall show that image of semi unit is also semi unit under the isomorphism between R and S. Consider that isomorphism between R and S, f : $R \longrightarrow S$, such that f(r) = s. Let a is semi unit in R. Then, 1 + va = wa ,for v, w $\in R$, therefore f (1 + va) = f (wa) $\Longrightarrow$ f(1) + f(v) f(a) = f(w) f(a)

$\Longrightarrow 1_s + v_s a_s = w_s a_s$, where 1_s is identity in S ,$v_s, w_s, a_s \in S$. This shows that a_s is semiunit in S and f maps semiunit of R to semiunit of S.

Now to check the edges, If $x, y \in R$ such that x + y $\in (S_u)_R$, semi units in R. Then, $f(x), f(y) \in S$, such that f(x) + f(y) = f(x + y) $\in (S_u)_S$,semi units in S. Hence, whenever R ,S are isomorphic then so is there semi unit graphs.

3.7 Lemma

Let S be a semi ring with identity with N_u is non-ideal.Then there may some distinct elements in N_u which are non-adjacent.

Proof Obvious from [Theorem 2.3]

3.8 Proposition

Let S be a semi ring and N_u is an ideal. If ,$a, b \in S$, $a \neq b$ then the length of the path between a to b is 1, 2, 3, 4, otherwise a is not adjacent with b, that is
$d(a, b) \in \{1, 2, 3, 4 \ or \ \infty\}$.
Proof We shall discuss three cases to prove the above theorem
1) If $a, b \in S_u$ then $a + b \in S_u$ or $a + b \neq S_u$. If $a + b \in S_u$ then d(a,b) = 1 . If $a, b \in S_u$ such that $a + b \notin S_u$ then 0 is adjacent to both a and b , so there is path a — 0 — b between a and b , therefore d(a,b) = 2.
2) If $a \in S_u$, $b \notin S_u$, If $a + b \in S_u$ then d(a,b) = 1.If there exist some $c \in S_u$ such that $b + c \in S_u$ then there is path b — c — 0 — a with d(a,b) = 3.
On the other hand, if for all $c \in S_u$ such that $b + c \notin S_u$, then d(a,b) = ∞.
Similarly, if $a \notin S_u$, $b \in S_u$, then the same situation arises.
3)If $a, b \in N_u$ such that $a + b \in S_u$ then d(a,b) = 1
If $a, b \in N_u$ such that $a + b \in N_u$, and there is some possibility to find $c, d \in S_u$ such that $a + c \in S_u$ and $b + d \in S_u$, then there is shortest path a — c — 0 — b — d with d(a,b) = 4.
Otherwise, if there is no path to connect a and b then $d(a, b) = \infty$

3.8.1 Examples

1) In every Semi field S, provided that it is not a field then every non-zero element is a unit hence semi unit in S, and S_u is closed under addition that is for all $a, b \in S_u = S - \{0\}$, there is $a + b \in S_u = S - \{0\}$.
so d (a, b) = 1 for all $a, b \in S$.
2) Consider a set S = {0,1,2,3,4,5} with binary operation + as maximum and × as minimum. Here 0 treats as zero of the set while 5 behaves as identity. This ,clearly makes a semi ring with non-zero identity. $S_u = \{5\}$ and $N_u = \{0, 1, 2, 3, 4\}$. Consider two vertices 1,4 there is no direct edge between them but if we take the paths 5 — 1 and 5 — 4 them there is path 1 — 5 — 4 , Hence d (1,4) = 2.
3) Consider $S = Z_6$ with usual addition and multiplication , where $S_u = \{1, 5\}$, if we consider the path between 0 and 3, than there is no edge in them but a path exists by taking 0 — 1 — 4 — 3 or 0 — 5 — 2 — 3, so d(a,b) = 3.
4) Consider a semi ring N_0 = {0,1,2,3,. . . . .} Here 1 is identity and only semi unit in this semi ring ,$d(1, i) = \infty$ for all $i \in N$.

3.9 Proposition

In finite local semi ring S with identity, the non-semi units makes k-ideal, therefore in semi unit graph $d(a, b) \in \{1, 2\}$.
Proof 1) If $a, b \in S_u$ such that $a + b \in S_u$, then d(a,b) = 1 .
If $a, b \in S_u$ such that $a + b \in N_u$ then there is path a — 0 — b ,so d(a,b)=2.
2) If $a \in S_u$, $b \in N_u$ then it must $a + b \in S_u$ .Hence d(a,b) = 1. Similarly, when $b \in S_u$, $a \in N_u$.
3) If $a, b \in N_u$ then $a + b \in N_u$ .There must exist a path a — u — b for some $u \in S_u$.Hence d(a,b) = 2.

3.10 Theorem

(a) If S is local semi ring with zero and O is the only non-semi unit (or a semi field with 0) than $diam\xi = sup\{d(x, y), x, y \in S\} = 1$.
(b) If S is finite local semi ring with non-zero identity and $|N_u| \geq 2$,
$then\ diam\xi = 2$.
Proof (a) We know that if S is local semi ring with O is the only non-semi unit than the ξ is complete (c.f. Theorem 3.1). Consider $x, y \in S_u = S - \{0\}$, then$x + y \in S_u$ that the path is x — y with size 1. Since, In semi ring the semi units are zero-sum free. Also there is path x — 0 — y with size 2. But smallest distance d(x,y) = 1 Hence, $diam\xi = 2$.
(b) Here $|N_u| \geq 2$, so if $a, b \in N_u$ than $a + b \in N_u$. But N_u is k-maximal ideal so $a \in N_u\ and\ u \in S_u$, there must $a + u, b + u \in S_u$. So by taking the path a — u — b , d (a, b) = 2, and therefore $diam\xi = 2$.

3.11 Theorem

In Local semi ring S such that $|S| \geq 3$
a) If $|S_u| \geq 2$, then $gr(\xi) \geq 3$
b) If $|S_u| < 2$, then $gr(\xi) = \infty$
Proof a) When $|S_u| \geq 2$, then there exist two or more than two semi units and we know that For all $x \in N_u and\ y, z \in S_u$
$x + y \in S_u\ and\ x + z \in S_u$
If $y + z \in S_u$ .So, there is cycle $z \longrightarrow x \longrightarrow y \longrightarrow z$ Hence girth (ξ) = 3. While if $y + z \notin S_u$, then for any $t \in N_u$,such that $x + t, y + t \in S_u$ then there is cycle $x \longrightarrow y \longrightarrow t \longrightarrow z \longrightarrow x$

So it is cycle of length 4 and girth is greater than 3.
b) When $|S_u| < 2$, so there is only one semi unit . Consider $x \in\ N_u\ and$ $u \in S_u\ so\ x + u \in S_u$,

Also, for every $x, y \in N_u so\ x + y \in N_u$, Therefore, there is no cycle. Hence $girth(\xi) = \infty$.

3.12 Lemma

Consider S is commutative semi ring with non-zero identity 1. If S_u is the set of all semi units and $2 \notin S_u$, then $\xi = \bar{\xi}$ $(closed\ graph)$.
Proof We have $1 + 1 = 2 \notin S_u$, i.e. there is no loop at 1. We will show that there is also no loop on any $x \in S$. Suppose on contrary$x + x = 2.x \in S_u$. It gives
1 + (2.x).r = (2.x).s , for some $r, s \in S$. $\implies 1 + 2.(x.r) = 2.(x.s) \implies$ $1 + 2.r_1 = 2.s_1$ where $r_1 = x.r, s_1 = x.s \in S$ which tells that 2 is semi unit , which is contradiction ,so $2.x \notin S_u$,Hence no loop in whole graph therefore $\xi = \bar{\xi}$.

3.13 Theorem

Let S be a semi ring with non-zero identity, then the semi unit graph ξ is a closed complete graph if and only if S is semi field with char(S) = 2.
Proof Suppose that the semi unit graph ξ is a closed complete graph. We will prove that S is semi field with char(S) = 2. Let $0 \neq r \in S$, the zero of S is adjacent to r, so every element is semi unit (by definition of semi unit graph) implies that $N_u = \{0\}$.
Before going to show that every non-zero element is unit, we shall prove that char(S) = 2. As $0 + 0 = 0 \notin S_u$, therefore there is no loop at 0. The graph is closed complete therefore there would be no loop on any vertex of graph. Hence the graph is simple. This implies that x + x $\notin S_u$, because 0 has no loop so all vertices must have no loops. As x + x $\in N_u = \{0\}$, therefore
2x = 0, for every x $\in S$ .This shows that char S = 2 .
Now, to check that every r is unit element. We have proven that every non-zero $r \in S$ is semi unit, and 1 + r.a = r.b for some $a, b \in S$ Adding r.a on both sides ,we get 1 + 2r.a = r(b + a) $\implies 1 = r.(b+a)$, so r is unit element too.Hence S is semi field with char(S) = 2.
Conversely, Suppose that S is semifield with char(S) = 2 therefore every non zero element is a unit so also semi unit and every semi unit must be zero sum free in semi field ,that shows for all different $x, y \in S_u, x+y \in S_u$ .Hence it is complete graph without any loop, as char (S) = 2 implies that the possibility of formulation of loop is ruled out so complete closed graph.

3.13.1 Remark

Each finite semi field is either a field or isomorphic to Boolean semi field (Corollary 5.9 [9]). Boolean field does not have Char (2) so we take fields as Char (2).

3.13.2 Example

If a is the root of this polynomial in GF (4), than the following is field of Char = (2)

+	0	1	a	1 + a
0	0	1	a	1 +a
1	1	0	1 + a	a
a	a	1 +a	0	1
1 + a	1 + a	a	1	0

×	0	1	a	1 + a
0	0	0	0	0
1	0	1	a	1 + a
a	0	a	1 + a	1
1 + a	0	1 + a	1	a

Its graph is complete closed semi unit graph as shown in fig (1a).

3.14 Theorem

Let R be a commutative semi ring with non-zero identity and S is additive subgroup with identity of R , then
(a) If $2 \notin S_u$, then the semi unit graph ξ of S is $|S_u|$ -regular graph
(b) If $2 \in S_u$, then for all x $\in S_u$, we have deg (x) = $|S_u|$ - 1 And for all x $\in N_u$ we have deg (x) = $|S_u|$.

Proof **(a)** Suppose $x \in S$, then we know in additive subgroup S with zero then S + x = S Now considering S_u is the set of all semi units u of S, there exist some $x_u \in S$ such that $x_u + x = u$. Clearly x_u is uniquely determined by u. If $2 \notin S_u$, then $x + x = 2.x \notin S_u$ by (lemma 3.10) . It tells that $x_u \neq x$, so x_u is adjacent in graph with x only, therefore we can define a mapping
$\theta : S_u \longrightarrow N_\xi(x)$, where$N_\xi(x)$ is set of neighborhood vertices .
such that $\theta(u) = x_u$, this function is clearly well defined and bijective therefore $|N_\xi(x)| = |S_u|$,that deg (x) = $|S_u|$, so the graph is $|S_u|$ -regular graph.
(b) : Here R be a commutative semi ring with non-zero identity and S with identity is additive subgroup of R, then S + x = S And so for every u $\in S_u$,there exists $x_u \in S$ such that $x_u + x = u$ Clearly x_u is uniquely determined by u. Now suppose that $2 \in S_u$ and x $\in N_u$ so that 2.x $\in N_u$ by (Theorem 2.3) . In this case, we have again $x_u \neq x$ and so x_u is not adjacent to x , therefore the previous observation (a) is still valid, which shows that
deg (x) = $|S_u|$.
Next suppose that $2 \in S_u$, and $x \in S_u$, then 2.x $\in S_u$ by (Proposition 2.2).

In this case we have $x_u \neq x$ for u $\neq 2x$ And $x_u = x$ for u $= 2x$. Now define a mapping
$\theta : S_u \longrightarrow N_\xi[x]$, (where$N_\xi[x]$ is set of neighborhood vertices including x itself) , such that $\theta(u) = x_u$ .This is well-defined and bijective therefore $deg(x) = |N_\xi[x]| - 1$,as loop cannot be considered in simple graphs.

3.14.1 Examples

1) Consider a semi ring B(5,1) ,where both operation of addition and multiplication has mod $5 - 1 = 4$.These operations are completely elaborate in the following tables.

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	0	1	2
2	2	3	0	1	2	3
3	3	0	1	2	3	0
4	0	1	2	3	0	1
5	1	2	3	0	1	2

×	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	0	1
2	0	2	0	2	0	2
3	0	3	2	1	0	3
4	0	0	0	0	0	0
5	0	1	2	3	0	1

Here we take out additive subgroup S = {0,1,2,3}, with identity 1 and S_u= {1,3} with $2 \notin S_u$,so its semi unit graph is 2-regular graph as in fig(1b).
2) Consider a semi ring B(5,2) ,where both operation of addition and multiplication has mod $5 - 2 = 3$.Here we take out additive subgroup S = {0,1,2}, with identity 1 and S_u= {1,2} with $2 \in S_u$ than deg (x) = 2 if
x $\in S_u$ and deg (x) = 1 $if x \notin S_u$ as shown in fig(1c).

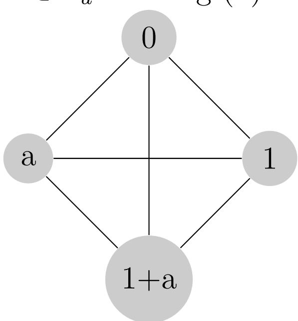


fig (1a)

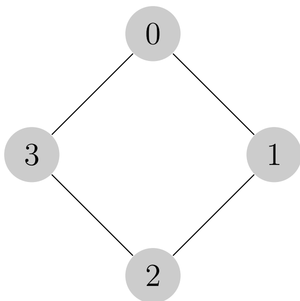
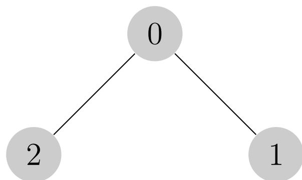


fig (1b)



fig(1c)

3.15 Theorem

Let S be a commutative semi ring and M be a Q- maximal ideal of S such that $|S/M| = 2$ and $2 \notin S_u$. Then $\xi(S)$ is complete bipartite graph.
Proof Let $V_1 = M$ and $V_2 = S - M$. Here M is Q-ideal therefore k-ideal, and $|S/M| = 2$ therefore $S/M = \{M, M + a\}$, for some $a \in Q$, is a semiring

with M is zero of S/M. Here M is k-maximal ideal so $N_u \subseteq M = V_1$ implies that $S_u \subseteq M + a = S - M = V_2$. First we show that $N_u = M$, Suppose on contrary that there exist some $x \in N_u \subseteq M$ and $x \in M + a$, implies that M = M + a, by definition of Q partitioning , which is a contradiction. Hence $N_u = M$, therefore we have $V(\xi) = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$, and V_1 and V_2 makes partition of $V(\xi)$ into two subsets.
It is clear that no pair of distinct elements of V_1 are adjacent.Now to prove that $\xi(S)$ is bipartite, we now only to show that no two elements of V_2 are adjacent. Then by assumption , there is some $a \in Q$ (a must be semi unit) such that $S = M \cup (M + a)$. Now for distinct,$x, y \in S - M = M + a$. Suppose on contrary that x and y are adjacent then x + y $\in S_u = S - M = M + a$ .therefore x + y = $(m_1 + a) + (m_2 + a) = m_1 + m_2 + 2.a$ =$m_3 + 2.a = m_4 \in M$ tells that semiunit x + y in M which is contradiction , so elements of V_2 are non-adjacent among themselves. Hence $\xi(S)$ is bipartite graph.
Now, we have to prove that $\xi(S)$ is complete bipartite. Let $x \in V_1$ and $y \in V_2$ If $x + y \notin V_2 = S_u = S - M$, then $x + y \in V_1 = M$, and M is Q-ideal therefore k-ideal therefore $y \in M$ which is a contradiction. Thus $x + y \in S_u$. So, x and y are adjacent. Therefore each vertex of V_1 is joined to each vertex of V_2 .Hence $\xi(S)$ is completely bipartite.

3.16 Proposition [5]

Let S be a partitioning semi ring, and let r $\in S_u$. Then r $\in$ J(S) if and only if, for every a $\in$ S, the element 1 + r.a is a semi-unit of S.

3.16.1 Example

Consider a set S = $\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, ..., 0\}$, This is semi ring with binary operation + = max and usual multiplication. Here only semi unit is identity 1, while all others are non-semi units. These non-semi units makes maximal ideal, with Q-partitioning .Here Q= 1, such that M, M+1 make partitioning, it makes complete bipartite graph ■

A semi ring S is Noetherian (respectively Artinian) if any non-empty set of k-Ideals of S has a maximal member (respectively minimal member) with respect to set inclusion. The definition is equivalent to the ascending chain condition (respectively descending chain condition) on k-Ideals of S.
Every finite Semi ring is Noetherian or artinian Semi ring. Here we are going to introduce our previous results into infinte semi rings .

3.17 Proposition[10]

Let S be an Artinian cancellative semi ring. Then
i. Every element of S is either a semi unit or a nilpotent element.
ii. S is a local semi ring.

3.18 Theorem

a) Let S be an Artinian cancellative semi ring with identity then is connected.
b) Let S be an Artinian cancellative semi ring, then for any non-semi unit x, deg (x) = $|S_u|$
c) Let S be an Artinian cancellative semi ring then for all semi units x, deg(x) = $|S|$ - 1
d) Let S be an Artinian cancellative semi ring then d(a,b) = 1 or 2 , for all $a, b \in S$
e) Let S be an Artinian cancellative semi ring than $diam(\xi) \leq 2$.
Proof Obvious by (Proposition 3.17) and previous results.

3.19 Example

Consider an artinian semi ring $Z^+ \cup \{0, \infty\}$. Here $S_u = \{1, 2, 3, ..., \infty\}$ and $N_u = \{0\}$. From fig we can easily check all the above mentioned results of Theorem 3.19 are valid. ■

The connectivity of a graph ξ , denoted by $k(\xi)$, is defined to be the maximum number of vertices, it is necessary to remove from ξ in order to produce a disconnected graph.

3.20 Theorem

If S is local semi ring then ξ is connected, while $k(\xi) = |S_u|$.
Proof Consider S is local semi ring then non-semi units make k-maximal ideal, and for every$x \in N_u$ and $u \in S_u$ implies that x + u $\in S_u$ So, there is connection between every non-semi unit and unit. To make the graph disconnected there must not link edge between non-semi units, so $k(\xi) = |S_u|$.

3.20.1 Corollary

If S is Artinian semi ring then ξ is connected, while $k(\xi) = |S_u|$.

3.20.2 Corollary

If S is finite local semi ring and $S_u = 1$, then ξ is connected (and complete) and $k(\xi) = |S_u| = 1$. ■

Gelfand semiring is defined in [2].It is a semiring S with identity 1, such that 1 + a is a unit for all $a \in S$.In Gelfand semiring the sum of two units is always a unit [13].

3.21 Lemma

In a Gelfand semi ring S ($|S| \geq 3$) with non-zero Identity along with unit elements ($U(S) \geq 2$) than the girth of semi unit graph is 3.
Proof if a,b are two units in S than the sum of two units in a Gelfand semi ring is also unit (*Prop* 4.50[2]) so this will create a cycle 0—a—b—0 .Hence the girth of its semiunit graph is 3 .

3.22 Theorem

In a local semi ring S, if $|S|$ ≥3 with $|S_u| = |N_u|$, then is Hamiltonian Graph.
Proof N_u is a k-ideal and so every semi unit is adjacent with every non semi unit. For any x ∈ S the degree d(x) = $|S_u| = |N_u|$, so if there exist non-adjacent vertices then for any non-adjacent vertices x, y ∈ N_u ,

$$\text{d(x) + d(y)} = |S_u| + |S_u| = |S| \text{ ....(1)}$$

Also if x, y ∈ S_u ,then d(x) + d(y) $\geq |S_u| + |S_u| = |S|$ ....(2)
Fromm (1) and (2) , we have d(x) + d(y) $\geq |S|$,therefore this makes Hamiltonian Graph [14].

Acknowledgements

I would like to thank the referee for helpful suggestions which have improved the paper.

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